TESTING AS A MEANS OF IMPROVING THE TEACHING OF HIGH SCHOOL MATHEMATICS.

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Testing, in the sense in which the word used in this article does not refer to the so-called standardized test, but to the ordinary class examination, or class test.

In a modern business every detail in the manufacture of an object is examined critically. Similarly, the teacher should employ a critical examination to measure the results of his teaching. Tests may be used to show the complexity of the work and to reveal definite problems for the teacher. Indeed, it is found that they present more problems than they solve. It may be said that the principal value and aim of all testing should be to improve the quality of teaching by means of an analytical exposition of its problems and difficulties. Opinion must be replaced by knowledge, guess work by evidence. This type of testing is fundamental in the progressive organization of a course.

Briefly stated the method might be as follows: First, some concrete aims are set up and agreed upon. Second, tests are designed and given, to establish to what extent the abilities to be developed in the pupil have been attained. Third methods of instruction are developed that are adapted to improve these abilities. Finally, suitable tests are devised to determine whether the course is actually helping the pupil to gain in these abilities in order to help him further if he is still found deficient.

This seems a very simple and sensible program. Following this program we shall first make in mathematics a list of the important concrete aims of the chapters, principles, or topics taught, such as the development of the abilities to manipulate the formal operations of algebra, to translate verbal problems into algebraic symbols, to grasp space relations, to classify and to generalize. Having selected our subject matter ac-

cording to our best judgment, the next step will be the designing of tests by which to measure the achievement of the pupils with this subject matter. The problem of constructing such tests is as yet only in the beginning stage.

That the ordinary class examination does not always establish the facts desired may be seen from the graphs in Fig. 1. They represent the grades of 50 pupils in a test given to three classes taught by three different teachers in a third-year course. Although classes and teachers vary in ability, the general shape of the graphs is about the same. They show that either the pupils do very well in the test, or they fail. It is hardly possible that in all classes the pupils were either very poor or very good, and we may assume that the teachers were at least better than the average. Evidently, the test gives no information about the poorer students, for the work seemed to be so hard that they were unable to show what they could do. Nor does it tell anything about the bright students who were able to

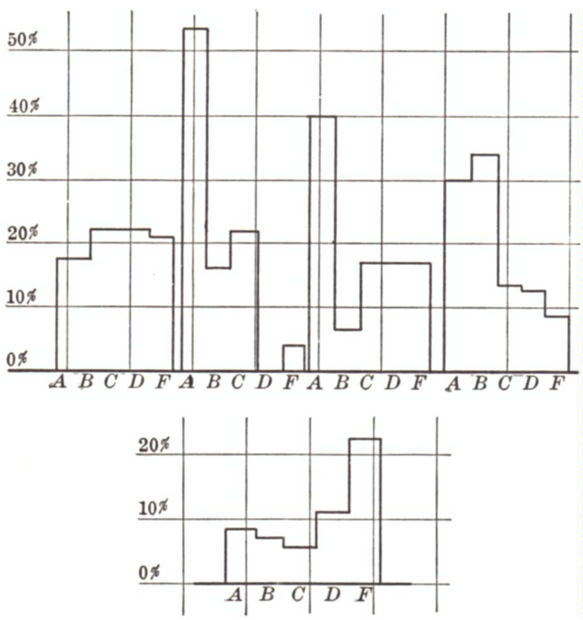


FIG. 1. Test in factoring—50 pupils—3 classes—Mathematics III., February 7, 1913.
Classes I., II. and III.

complete the test in less than the required time and therefore could not show what they were really able to accomplish. A more suitable test must contain enough work of the simple type for the slow pupil. It must also contain some difficult work which only some of the best pupils can do. The graphs in Fig. 2 show that 13 per cent. of the pupils failed. Apparently

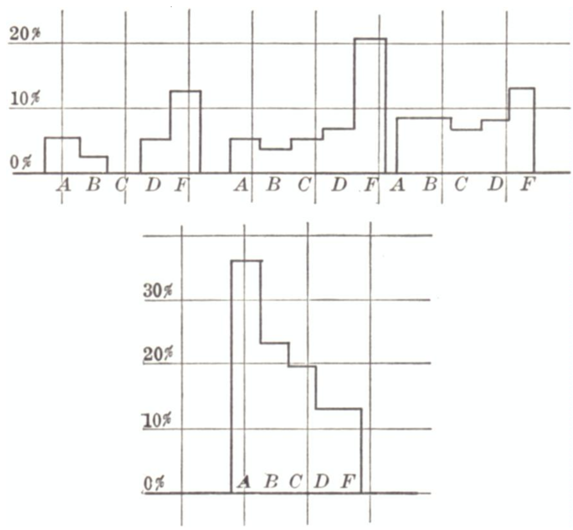


FIG. 2. Test V. Mathematics I., Chapters VI. and VII. Four classes.

in this test the slow pupils were able to show that they could do some of the work.

However, in this as well as in the first test, a large percentage of pupils failed to gain real mastery of the work in which they were being tested, and needed further teaching. A more detailed study must be made to reveal what the nature and extent of this instruction should be.

To measure the results in a test each problem is divided into the component elements which enter in the solution. Each problem is analyzed as to the processes it aims to test. A score of 1 is then given for each process performed correctly. If not correct it is scored 0. Attainment it the ratio of the actual score made to the largest possible score. Thus, if a process is performed correctly by 76 pupils in a group of 80, the ratio 76/80 = .95 is the attainment.

Before the test is given, a key for scoring the various processes is worked out. The following are a few examples showing how this may be done for some of the typical work in algebra and geometry.

OPERATIONS.

1. Multiplication of monomials is one of the simplest problems in algebra. It is also one of the most fundamental processes, and must therefore be mastered by all pupils. The problem involves three steps, the determination of the sign, the arithmetical product, and the literal product.

Each of these is scored separately. For example, in $(-12m^5)(-16mr^3)$,

the sign is $+$	1	Therefore, the score is 3.
the arithmetical product is 192.............	1	
and the literal product is m^6r^3.............	1	

It must be remembered that the score does not indicate a measure of the degree of difficulty or a value of the step.

2. The *division of monomials* involves the same three steps. Thus in $-6a^2cx^2y \div 2axy$, the sign

is $-$	1	Therefore, the score is 3.
the arithmetical quotient is 3................	1	
the literal quotient is acx.....................	1	

3. The component parts of *multiplication of a polynomial by a monomial* are the multiplications of a number of monomials. In the following example this occurs three times. Therefore the score is 3, meaning that three operations have been performed correctly:

$(3a^3 - 2a^2 + 5a)(-2a^2) = -6a^5$	1	
$+4a^4$	1	
$-10a^3$	1	

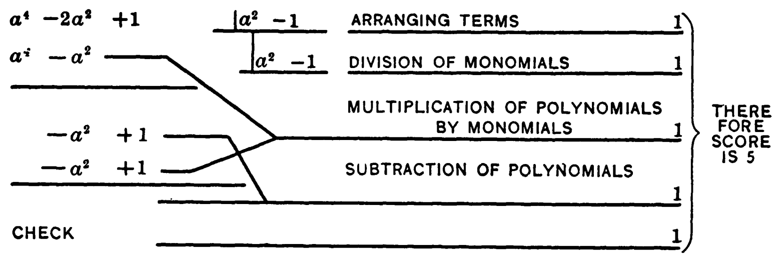
4. *In the multiplication of polynomials* we have the multiplication of a polynomial by a monomial, and the addition of the resulting polynomials.

Accordingly, the product $(a^3 - a^2 - 1)(a + 1)$ is scored as follows:

$a^4 - a^3 - a$ 1
$\quad + a^3 - a^2 - 1$ 1
$a^4 \quad - a^2 - a - 1$ 1

Therefore, the score is 3.

5. *In the division of polynomials* we have the following steps.



Similarly we may now show how to analyze the solution of equations.

EQUATIONS.

6. *Solution of a linear equation:*

$$y - \frac{2y-4}{3} = \frac{2(5-y)+7}{9}.$$

The processes are

1. Clearing of fractions:
   $9y - 3(2y - 4) = 2(5 - y) + 7$ 1
2. Multiplying polynomials by monomials:
   $9y - 6y + 12 \quad = 10 - 2y + 7$ 1
3. Combining similar terms:
   $3y + 12 \quad = 17 - 2y$ 1
4. Adding to, and subtracting from both members:
   $5y \quad = 5$ 1
5. Dividing both members:
   $y \quad = 1$ 1
6. Checking in the original equation:
   $\left\{\begin{array}{l} 1 + 2/3 = (8 + 7)/9 \\ 1\,2/3 = 1\,2/3 \end{array}\right\}$ 1

Therefore, the score is 6.

7. *Solution of a quadratic equation.*

$4x^2 + x - 39 = 0$

$$\left.\begin{array}{lll}
(4x+13)\ (x-3)=0 & \text{two factors} \ldots\ldots\ldots\ldots & 2\\
\left.\begin{array}{l}4x+13=0\\ x-3=0\end{array}\right\} & \text{two linear equations deducted} \ldots & 2\\
\left.\begin{array}{l}x=-13/4\\ x=3\end{array}\right\} & \text{two equations solved} \ldots\ldots\ldots\ldots & 2
\end{array}\right\}\ \text{Therefore, the score is 6.}$$

These scores are used when the process of solving the equation is first taught. When these equations occur later the complete solution may be considered as one operation.

8. *Simultaneous equations.*

(*a*) Solution by graph:

$$\left.\begin{array}{lr}
\text{1. Make tables} \ldots\ldots\ldots\ldots\ldots\ldots & 2\\
\text{2. Draw axes, and select units} \ldots\ldots\ldots & 2\\
\text{3. Plot points, and graph lines} \ldots\ldots\ldots & 2\\
\text{4. Solve by finding point of intersection} \ldots & 1
\end{array}\right\}7$$

(*b*) Solution by addition or subtraction:

$$\left.\begin{array}{lr}
3m+7n=34 & \\
7m+8n=46 \quad \text{multiply} & \\
21m+49n=238 \ldots\ldots\ldots\ldots\ldots\ldots & 1\\
21m+24m=138 \quad \text{subtract} \ldots\ldots\ldots\ldots & 1\\
25n=100 \ldots\ldots\ldots\ldots\ldots\ldots & 1\\
n=4 \quad \text{solve} \ldots\ldots\ldots\ldots\ldots & 1\\
3m+28=34 \quad \text{substitute} \ldots\ldots\ldots\ldots & 1\\
\left.\begin{array}{l}3m=6\\ m=2\end{array}\right\}\ \text{solve} \ldots\ldots\ldots\ldots\ldots & 1\\
(m,n)=(2,4) \quad \text{results} \ldots\ldots\ldots\ldots & 1
\end{array}\right\}\ \text{Therefore, the score is 7.}$$

Verbal Problems.

The verbal problems of algebra are subdivided into a similar scheme.

9. *Motion problem.*

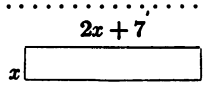
A courier who travels 6 miles an hour is followed after 2 hours by a second courier who travels $7\frac{1}{2}$ miles an hour. In how many hours will the second courier overtake the first. The following processes are involved.

	t	r	d		
1) Method:				1	The score for the whole problem is 5.
2) Data for one courier:	x	6	$6x$	1	
3) Data for the second courier:	$x-2$	$7\frac{1}{2}$	$7\frac{1}{2}(x-2)$	1	
4) Stating the equation: $6x = 7\frac{1}{2}(x-2)$				1	
5) Solving the equation: $6x = \frac{15x-30}{2}$, $x = 10$				1	

10. *Geometric problem solved algebraically.*

One side of a rectangle is 7 feet longer than twice the other, the perimeter being 54 feet. Make a sketch of the rectangle and find the length of the sides.

1. Let x be the number of feet in one side. Then $2x$ $7+$ is the number of feet in the other side. Figure: } data.....1



2. State equation: $2x + 4x + 14 = 54$.......... 1
3. Combine: $6x + 14 = 54$..........
4. Subtract: $6x = 40$............... } solution...1
5. Divide: $x = 6\,2/3$................
6. Other solutions: $2x + 7 = 20\,1/3$............ 1
7. Check: $6\,2/3 + 6\,2/3 + 20\,1/3 + 20\,1/3 = 54$...1

Therefore the score is 5.

11. *Problems In Indirect Measurement.*

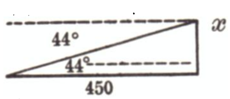


Figure and notation...... 1
Alternate int. angles are equal 1
$\tan 44° = x/450$ Choice of function........... 1
$x = 450 \tan 44$ Solution 1
$x = 450\,(.966)$ Use of table............ 1
$x = 434.7$ Arithmetical computation..... 1

Therefore, the score is 6.

Geometric Problems and Theorems.

In the scoring of *theorems* or *problems* in geometry the following general method is used:

1. Figure and helping lines............................ 1
2. Given, to prove 1
3. Explain how helping lines are drawn 1
4. Congruent triangles 4
5. A single statement giving a correct inference with the correct reason for it.............................. 1

The following proof illustrates the method.

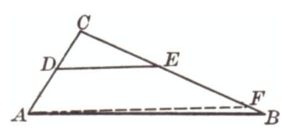
12. *Use of the indirect method of proof.*

Theorem: Two lines that cut two intersecting lines and make the corresponding segments of the given lines proportional are parallel.

Given: DE and AB cut by CA and CB, making $CD/DA = CE/EB$ ....

To prove: $DE \parallel$ to AB .... } 1

Figure:



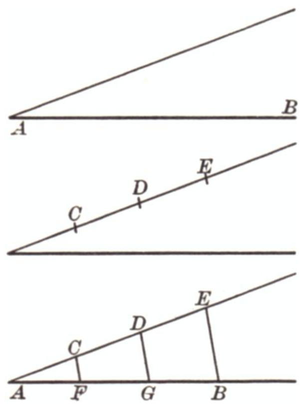
Proof: Assume DE not parallel to AB.................. 1
Draw AF parallel to DE......................... 1
Then $CD/CA = CE/EF$....................... 1
But $CD/DA = CE/EB$........................ 1
$\therefore$ $CE/EB = CE/EF$ 1
$\therefore$ $CE \cdot EF = CE \cdot EB$ 1
$\therefore$ $EF = EB$ 1
This is impossible 1
$\therefore$ The assumption is wrong 1
CF parallel DG parallel EB..................... 1

13. *A construction with ruler and compass.*

To divide a segment into three equal parts.

Given: A segment AB
Required: To divide AB into 3 equal parts 1

Construction:



Draw helping line......... 1
Lay off 3 equal segments... 1
Draw parallels 1

Proof: $AC = CD = ED$ 1
CF parallel DG parallel EB 1
$AF = FG = GB$ 1

To obtain a clear understanding of the study of tests we may examine in detail the results of one of the tests given to a class in First-Year Mathematics. None of the problems in this test were taken from the text book used by the pupils. The test is as follows.

Test VIII., Mathematics I.

I. Change to the simplest form by collecting terms

$$6x^2 + 8 - 3x^2 - 5 - 9x^2 - 7$$

$$3g + f - 8g - 6f - 7g - 5f$$

$$-8t^3 - 4t + 6t^2 - 3t + 2t^3 = 9t$$

$$-2ab - 3g + 7ab + 7g + 6ab \div 4g$$

II. If $c = 4$ and $f = 2$ what does $2c^3 - 3f$ equal?
if $a = 3$ and $b = 2$ what does $3ab + ab^2$ equal?
if $x = 3$ and $y = 4$ what does $xy^2 - 2xy$ equal?
if $r = 2$ and $s = 4$ what does $r^2 + 3r^2s$ equal?

III. From $2d + 13f$ take $7d + 14f - 6g$
Take $9h + 14k$ from $7h - 3k + 8s$
Subtract $6m - 11n + 13p$ from $m + 3n + 7p$

IV. Perform the indicated operations and reduce to the simplest form:

$3(7y) =$ $\qquad$ $(4x)(-3xy^3) =$

$(2a)(4ab^2) =$ $\qquad$ $(a^3)(-3a)(-2a) =$

$2/3$ of $9m =$ $\qquad$ $(-3xy^3)\,4 =$

V. Simplify $7(5x + 9)$; $5(8x - 4)$; $-6(5x + 2)$; $-9(4x - 6)$; $7(-6x - 4)$; $-8(-4x - 6)$.

VI. Perform the indicated operations and reduce to the simplest form:

$$\frac{12n}{4} = \qquad \frac{7a}{15} \div \frac{7a^2}{20} =$$

$$6c^3 \div 2c^2 = \qquad \frac{-12x^2y^2\cdot(x-2)}{-3x^2y^2} =$$

$$\frac{-8a^2b}{4a^2} =$$

$$\frac{4x^4}{5} \div 2x^2 =$$

VII. Perform the indicated operations and reduce to the simples form:

$$(2a^2 + 7a - 9)\quad(5a - 1) =$$

$$\frac{18m^2n - 27mn^2}{9mn} =$$

$$(x^3 - x^2 - 4x + 4) \div (x^2 - 3x + 2)$$

VIII. Show graphically each of the following statements to be true. Explain your drawing.

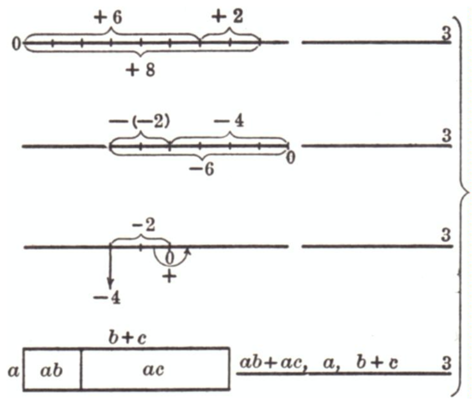
$$(+8) + (-2) = +6;\ (-6) - (-2) = -4;$$
$$(-2)\quad(-4) = +8;\ ab + ac = a(b + c)$$

The first step in the study is to make the key for scoring. To avoid needless repetition only part of the key used in scoring the test above is shown as follows.

Key for Test VIII., Mathematics I.

	Answer and scoring	Points	Total
I.	$-6x^2-4$ Combining terms	2	Total score 9.
II.	$2\cdot 16-24=+8$ Substitute, multiply, combine	3	Total score: 12.
III.	$-5d-f+6g$ Each term is scored 1, if right	3	Total score 9.
IV.	$21y$ Numerical product	1	Total score: 13.
	$8a^2b^2$ Numerical product, literal product.	2	
	$6m$ Numerical product	1	
	$-12x^2y^2$ Sign, numerical product, literal product	3	
	$+6a^5$	3	
	$+81x^4y^{12}$	3	
V.	$35x+63$. Each term scores 1, if correct.	2	Total score: 12.
VI.	$3n$ Arithmetical quotient	1	Total score: 15.
	$3c$ Numerical part, literal part.	2	
	$-2b$ Sign, numerical part, literal part.	3	
	$2/5\,x^2$ Invert, multiply, reduce	3	
	$4/3a$ Invert, multiply, reduce	3	
	$+4(x-2)$ Sign, numerical part, literal part	3	
VII.	$10a^3+35a^2-45a+$ Polynomial by monominal $-2a^2-7a+9$ Polynomial by monomial $10a^3+33a^2-52a+9$ *Combine*	3	Total score: 8.
	$\dfrac{9mn(2m-3n)}{9mn}$ $2m-3n$ Factor, reduce	2	
	x^3-3x^2+2x $2x^2-6x+4$ $x+2$ Divide, multiply, subtract	3	

VIII.
Total
score: 12.



It appears that ninety different operations were to be performed in this test.

Having scored each process of every problem for every paper by following the directions of the key, we find for every process the ratio of the number of scores actually made to the ratio of the total number possible. Table I., below, gives these ratios for every problem for each of the four classes. It shows wide variation in the result of the classes, the largest interval being 32 and the smallest 11,

TABLE I.

Problems.	1	2	3	4	5	6	7	8
Class I.	96	85	94	94	98	88	78	81
Class II.	83	71	85	79•	98	64	64	59
Class III.	64	86	67	75	87	66	56	—
Class IV.	92	69	77	81	90	71	56	62
All classes.	88	71	84	86	96	83	64	69
Variations.	$\boxed{32}$	17	27	19	$\boxed{11}$	24	22	22

The table should be read as follows. Class I. in problem I. had an attainment of 96, etc.

Notice that the results in the various classes show wide variations, the largest being in problem 1 and the smallest in

problem 5. Class I. makes the best showing and class III. the poorest. An explanation of this difference may be found by securing collaborative evidence from the teachers. With a little more teaching of problems like 2, 7, and 8 class I. seems to be prepared to go on with the course, while class III. might as well study the whole chapter a second time.

If the problems are now ranked in order of attainment, table II. it is seen that the variations in *rank* are small.

TABLE II.

Problems.	1	2	3	4	5	6	7	8
Ranks for class I.	2	6	3	3	1	5	8	7
" " " II.	3	5	$\boxed{2}$	4	1	6	6	8
" " " III.	$\boxed{6}$	$\boxed{2}$	4	3	1	5	7	8
" " " IV.	1	6	4	3	2	5	8	7
" " all classes	2	6	4	3	1	5	8	7

Thus, problem 5 was very easy for all classes and problem 2 was hard. The remaining problems very nearly kept their positions. The table shows a uniformity in the results of the teaching in the four classes, because the ranks of the problems are practically the same although the classes are apparently not of equal ability.

Of special interest are the three exceptions marked with squares. When the teacher of class II. was asked why his class did so well with problem 3, he said that former experience had taught him that subtraction of polynomials is a difficult process. Therefore he placed great emphasis on this operation in his teaching. He apparently accomplished good results because in the subtraction of polynomials which enters also in problem 7, as part of the long division process, his class was superior to all others, the ratios for this process being respectively 54 per cent., 74 per cent., 38 per cent., 39 per cent. This also brings out the fact that a process is more difficult when it occurs in a new situation than when it occurs alone, —the ratios for this particular process in problem 3 being considerably higher, 94, 85, 67, and 77.

Class III. did not succeed well with problem 1. The

teacher's opinion was that the word "*collecting*" was new to the pupils, the word "*combining*" being used in class. Sometimes the use of an unfamiliar word in a test confuses the pupils, and the results do not indicate at all to what extent they are really able to work the problem.

Further evidence of their ability to solve this problem may be found by studying to what extent the pupils are successful when the same operation occurs elsewhere. For example, terms are to be collected in the third step in problem 7. The ratios for this step for the various classes are 81 per cent., 53 per cent., 23 per cent., and 60 per cent., showing that class III. is deficient in this process even when no instructions as to "collecting" or "combining" are given. In fact, an examination of the papers of the pupils verifies that they did not understand the process, for such mistakes as $a^2 + a^2 = a^4$ were frequent.

On this basis of the results of this study the teacher during the next class hour retaught the process, and similar errors were carefully corrected in the later part of the course.

That this class can be trained with careful teaching to do as good work as the others, or even better work, is seen from the fact that they surpassed all other classes in problem 2.

By arranging the various processes in the test in order of attainment some interesting comparisons can be made. The order is as follows:

TABLE III.

Processes.		Attainment
1. Multiplication of a binomial by a monomial		93
2. Graphical addition of positive and negative numbers		92
3. Combining similar terms		84
4. Subtraction of polynomials		84
5. Multiplication of monomials		83
sign	80	
numerical product	86	
literal product	80	
6. Reduction of fractions		78
sign	72	
numerical part	80	
literal part	78	
7. Multiplication of two polynomials		73
product of monomials	80	
adding polynomials	50	

8. Evaluation			69
	substitution	72	
	product of positive and negative numbers	69	
	combining terms	67	
9. Multiplying positive and negative numbers graphically			65
10. Dividing by a fraction			61
	inversion	67	
	multiplication	62	
	reduction	55	
11. Graphical subtraction			61
12. Division of polynomials			56
	division of monomials	64	
	product of polynomial by monomial	54	
	subtraction of polynomials	50	

The table shows that best results were attained with multiplying binomials by monomials, and the poorest in adding and subtracting polynomials as shown in 7 and 12. Hence, the latter need to be emphasized and used in many situations before they are really understood.

Graphical addition and subtraction were given as experimental problems. These processes are ordinarily used for illustrative purposes and not as an aim in themselves. It is not intended to drill pupils to be able to reproduce them. Yet one of them was retained as well as anything given in the test; the other proved to be difficult.

The results show uniformly that a process, when mastered alone, is not sufficiently understood until it has been mastered in different situations. For example, the product of monomials shows an attainment of 88; in process 7 it shows 80, in evaluation 69, in division 62. Similarly, for reduction of fractions we have an attainment of 78, in process 10 it shows only 55. For combining terms we have 84, but in evaluation only 67. For multiplication of polynomials by monomials we find an attainment of 80 in process 7, and 54 in process 12.

Subtraction of polynomials: alone 84 per cent., in long division 50 per cent. Multiplication of polynomials by a monomial: in multiplication of polynomials 80 per cent.; in long division 54 per cent.

It is evident that when a new process is taught, the previous processes involved offer new difficulties, and deserve considerable special attention. A similar situation is met when

arithmetical computations occur in an algebraic problem. Errors such as $2x \times 3x = 5x^2$ are common. Certainly the pupil knows that $2 \times 3 = 6$ but he must now concentrate on two processes at the same time, and he must learn to do both correctly. A review or drill in arithmetic alone will not help him to avoid such mistakes.

The process of long division standing at the bottom of the list, is a type of problem frequently entirely omitted from a first-year course. Since it serves as an excellent check on the extent to which the more fundamental processes are actually retained and carried over, it seems unwise to omit the teaching of long division of polynomials.

The test here recorded in full shows that an analytical exposition reveals the difficult parts involved in an operation or in a problem. This information may then be used to teach the problem again with an improved technique or in different situations. Moreover, similar records kept of all tests would be valuable to the teacher when he gives the course a second time and ultimately such systematic testing must lead to a well-organized course.